

Reduced Complexity Sum-Product Algorithm for Decoding Network Codes and In-Network Function Computation

Anindya Gupta and B. Sundar Rajan

Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, India

Email: {anindya.g, bsrajan}@ece.iisc.ernet.in

Abstract—While the capacity, feasibility and methods to obtain codes for network coding problems are well studied, the decoding procedure and complexity have not garnered much attention. In this work, we pose the decoding problem at a sink node in a network as a marginalize a product function (MPF) problem over a Boolean semiring and use the sum-product (SP) algorithm on a suitably constructed factor graph to perform iterative decoding. We use *traceback* to reduce the number of operations required for SP decoding at sink node with general demands and obtain the number of operations required for decoding using SP algorithm with and without traceback. For sinks demanding all messages, we define *fast decodability* of a network code and identify a sufficient condition for the same. Next, we consider the in-network function computation problem wherein the sink nodes do not demand the source messages, but are only interested in computing a function of the messages. We present an MPF formulation for function computation at the sink nodes in this setting and use the SP algorithm to obtain the value of the demanded function. The proposed method can be used for both linear and nonlinear as well as scalar and vector codes for both decoding of messages in a network coding problem and computing linear and nonlinear functions in an in-network function computation problem.

Index Terms—Network Coding, Decoding, Sum-Product Algorithm, Traceback, In-network Function Computation.

I. INTRODUCTION

In contemporary communication networks, the nodes perform only routing, i.e., they copy the data on incoming links to the outgoing links. In order to transmit messages generated simultaneously from multiple sources to multiple sinks the network may need to be used multiple times. This limits the throughput of the network and increases the time delay too. Network coding is known to circumvent these problems [1]. In network coding intermediate nodes in a network are permitted to perform coding operations, i.e., encode data received on the incoming links and then transmit it on the outgoing links (each outgoing link can get differently encoded data), the throughput of the network increases. Thus, network coding subsumes routing. For example, consider the butterfly network [1] of Fig. 1 wherein each link can carry one bit per link use, source node S generates bits b_1 and b_2 , and both sink nodes T_1 and T_2 demand both source bits. With routing only, two uses of link $V_3 - V_4$ are required while with network coding only one.

This is an example of single-source multi-sink linear multicast network coding, wherein there is a single source (S),

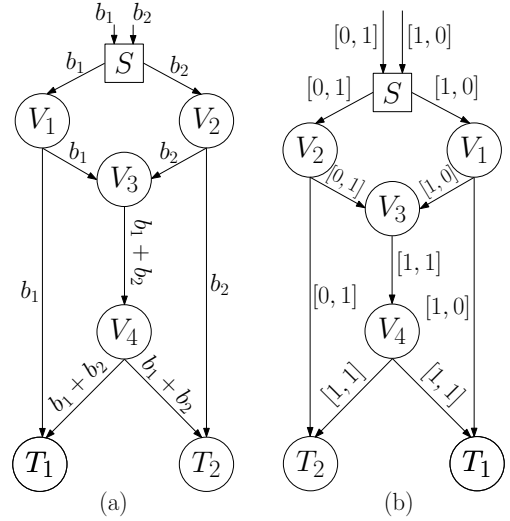


Fig. 1. The butterfly network: (a) A network code and (b) its global encoding vectors.

generating a finite number of messages, (x_1, x_2) , and multiple sinks, each demanding all the source messages and the encoding operations at all nodes are linear. In general, there may be several source nodes, each generating a different number of source messages, and several sink nodes, each demanding only a subset, and not necessarily all, of the source messages. Decoding at sink nodes with such general demands is studied in this paper.

A. Notations and Preliminaries

We represent a network by a finite directed acyclic graph $\mathcal{N} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of vertices or nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed links or edges between nodes. All links are assumed to be error-free. Let F denote a q -ary finite field. The set $\{1, 2, \dots, n\}$ is denoted by $[n]$. The network has J sources, $S_j, j \in [J]$, and K sinks, $T_k, k \in [K]$. The source S_j generates ω_j messages for all $j \in [J]$. Let $\omega = \sum_{j=1}^J \omega_j$ be the total number of source messages. The ω -tuple of source messages is denoted by $x_{[\omega]} = (x_1, x_2, \dots, x_\omega)$, where $x_i \in F$ for all $i \in [\omega]$. By $\mathbf{x} = (x_1, \dots, x_\omega)^T$ we denote the column vector of the source messages. The demand of the k^{th} sink node is denoted by $D_k \subseteq [\omega]$. Given a set $I = \{i_1, \dots, i_l\} \subseteq [\omega]$, let $x_I = (x_{i_1}, \dots, x_{i_l})$, i.e., $x_{[\omega]}$ restricted to I . For disjoint

subsets I and J of $[\omega]$, we do not differentiate between (x_I, x_J) and $x_{I \cup J}$. For a multi-variable binary-valued function $f(x_1, \dots, x_\omega)$, the subset of F^ω whose elements are mapped to 1 by $f(x_1, \dots, x_\omega)$ is called its support and is denoted by $\text{supt}(f(x_{[\omega]}))$ and $\text{supt}_I(f(x_{[\omega]}))$ denotes the $|I|$ -tuples in the support restricted to I . A source message is denoted by edges without any originating node and terminating at a source node. Data on a link $e \in \mathcal{E}$ is denoted by y_e .

A network code is a set of coding operations to be performed at each node such that the requisite source messages can be faithfully reproduced at the sink nodes. It can be specified using either local or global description [1]. The former specifies the data on a particular outgoing edge as a function of data on the incoming edges while the latter specifies the data on a particular outgoing edge as a function of source messages. Throughout the paper we use global description for our purposes.

Definition 1 (Global Description of a network code [1]):

An ω -dimensional network code on an acyclic network over a field F consists of $|E|$ global encoding maps $\tilde{f}_e : F^\omega \rightarrow F$ for all $e \in E$, i.e., $\tilde{f}_e(\mathbf{x}) = y_e$.

Let $e_i, i = 1, \dots, \omega$, be the incoming edges at the source, then $y_{e_i} = x_i$.

When the intermediate nodes perform only linear encoding operations, the resulting network code is said to be a linear network code (LNC).

Definition 2 (Global Description of an LNC [1]): An ω -dimensional LNC on an acyclic network over a field F consists of $|E|$ number of $1 \times \omega$ global encoding vectors \mathbf{f}_e for all $e \in E$ such that $\mathbf{f}_e \cdot \mathbf{x} = y_e$.

The global encoding vectors for the incoming edges at the source are standard basis vectors for the vector space F^ω . The global encoding vectors of the LNC for butterfly network is given in Fig. 1(b).

Hereafter we assume that the network is feasible, i.e., demands of all sink nodes can be met using network coding, and the global description of a network code (linear or nonlinear) is available at the sink nodes. If a sink node demands $\omega' (\leq \omega)$ source messages, it will have at least ω' incoming edges. The decoding problem is to reproduce the desired source messages from the coded data received at the incoming edges. Thus, decoding amounts to solving a set of at least ω' simultaneous equations (linear or nonlinear) in ω unknowns for a specified set of ω' unknowns. Hence, the global description of a network code is more useful for decoding.

While decoding of nonlinear network codes has not been studied, the common technique used for decoding an LNC for multicast networks is to perform Gaussian elimination [2], [3], which requires $\mathcal{O}(\omega^3)$ operations, followed by backward substitution, which requires $\mathcal{O}(\omega^2)$ operations [4]. This is not recommendable when the number of equations (incoming coded messages) and/or variables (source messages) is very large. In such cases, iterative methods are used. Convergence and initial guess are some issues that arise while using iterative methods [5].

We propose to use the sum-product (SP) algorithm to perform iterative decoding at the sinks. A similar scheme for decoding multicast network codes using factor graphs [6] was

studied in [7] in which the authors considered the case of LNCs. The problems associated with the proposed decoding scheme in [7] are:

- To construct the factor graph, full knowledge of network topology is assumed at the sinks which is impractical if the network topology changes. For a particular sink node (say T), the factor graph constructed will have $\omega + |E|$ variable nodes and $|E| + |In(T)|$ factor nodes, where $In(T)$ is the set of incoming edges at node T .
- Complete knowledge of local encoding matrix [1] of each node is assumed at the sinks which again is impractical since local encoding matrix for different nodes will have different dimensions and hence variable number of overhead bits will be required to communicate to downstream nodes which will incur huge overhead.

We also point out that the motivating examples, viz., Examples 1 and 4, given in [7] for which the proposed decoding method claims to exploit the network topology admits a simple routing solution and no network coding is required to achieve maximum throughput. Solving a system of linear equations in Boolean variables is also studied in [8, Ch. 18].

B. Contributions and Organization

The contributions and organization of this paper are as follows:

- In Section III-A, we pose the problem of decoding of linear and nonlinear network codes as a *marginalize a product function problem* (MPF) and construct a factor graph using the global description of network codes. For a particular sink node, the constructed graph will have fewer vertices than in [7] and hence the number of messages and operations performed will also be fewer. Unlike in [7], our scheme requires only the knowledge of global encoding maps/vectors of incoming edges at a sink node and not the entire network structure and coding operation performed at each node.
- In Section III-B, we utilize *traceback* [9] instead of running the multiple-vertex version of the SP algorithm which results in reduction in the number of operations. Application and advantage of using traceback over multiple-vertex SP algorithm for decoding at sinks with general demand is demonstrated.
- We discuss the utility and the computational complexity of the proposed technique in Section IV. We give the number of semiring operations required to perform single- and all-vertex SP algorithm for a class of MPF problem where we are interested not in the marginal function at a particular vertex but in the values of the variables/arguments in the local domain of that vertex that causes that marginal function to attain certain value in the semiring, for example, maximum in a max-sum or max-product semiring or minimum in a min-sum or min-product semiring. We call such problems as arg-MPF problems and refer to the application of the SP algorithm to such problems as the arg-SP algorithm. We show that the number of semiring operations required in performing single-vertex arg-SP algorithm with traceback is strictly

TABLE I
NUMBER OF SEMIRING OPERATIONS

	Sum-Product Algorithm	arg-Sum-Product Algorithm
Single-vertex	$\sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e - \sum_{v \in V} q_v$	$\sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e - \sum_{v \in V} q_v + q_r - 1$
All-vertex	$\sum_{z \in Z} (4d_z - 5)q_z + 2 \sum_{w \in W} q_w - 2 \sum_{e \in E} q_e$	$\sum_{z \in Z} (4d_z - 5)q_z + 2 \sum_{w \in W} q_w - 2 \sum_{e \in E} q_e + \sum_{z \in Z} q_z - Z $
Single-vertex with Traceback	Not Applicable	$\sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e + \sum_{w \in W} q_w - Z $

Note: V is the set of variable nodes, W is the set of factor nodes, $Z = V \cup W$, E is the set of edges, $G = (Z, E)$ is the factor graph, r is the chosen root node in G , and $a_r = 1$ if $r \in W$ and 0 otherwise. (See Section IV for a complete discussion.)

less than that of all-vertex SP algorithm (see Table I for a comparison). Hence, the decoding complexity of a network code using SP decoding with traceback is strictly less than that without using traceback. For sink nodes which demand all the source messages, the notion of *fast decodable network codes* is defined and a sufficient condition for the same is identified.

- In Section V, we consider the in-network function computation problem wherein the sink nodes demand a function of source messages. A network code for such a problem ensures computation of the value of the demanded function at a sink node given the coded messages on its incoming edges and not the reproduction of the values of the arguments of the function. Thus, multiple message vectors may evaluate to the same incoming coded messages and the demanded function value. In Section V-B, we show that obtaining one such message vector can be posed as an MPF problem and that obtaining it suffices for computation of the demanded function. Subsequently, we give a method to construct a factor graph for each sink node and use the SP algorithm to solve the MPF problem.

We present a brief overview of the SP algorithm in Section II. Preliminaries of in-network function computation are given in Section V-A. We conclude the paper with a discussion on scope for further work in Section VI.

II. THE SUM-PRODUCT ALGORITHM AND FACTOR GRAPHS

In this section, we review the computational problem called the MPF problem and specify how SP algorithm can be used to efficiently solve such problems. An equivalent method to efficiently solve MPF problems is given in [10] and is called the *generalized distributive law* (GDL) or the *junction tree algorithm*. The simplest example of SP algorithm offering computational advantage is the distributive law on real numbers, $a \cdot (b + c) = a \cdot b + a \cdot c$; the left hand side of the equation requires fewer operation than the right hand side. Generalization of addition and multiplication is what is exploited by the SP (or the junction tree) algorithm in different MPF problems. The mathematical structure in which these operations are defined is known as the commutative semiring [10].

Definition 3: A commutative semiring is a set R , together with two binary operations “+” (*addition or sum*) and “.”

(*multiplication or product*), which satisfy the following axioms:

- 1) The operation “+” satisfies closure, associative, and commutative properties; and there exists an element “0” (*additive identity*) such that $r + 0 = r$ for all $r \in R$.
- 2) The operation “.” satisfies closure, associative, and commutative properties; and there exists an element “1” (*multiplicative identity*) such that $r \cdot 1 = r$ for all $r \in R$.
- 3) The operation “.” *distributes* over “+”, i.e., $r_1 \cdot r_2 + r_1 \cdot r_3 = r_1 \cdot (r_2 + r_3)$ for all $r_1, r_2, r_3 \in R$

Different semirings are used for different MPF problem, each with a different notion of “+” and “.”. Some examples are listed below.

- 1) Application of the SP algorithm to the discrete Fourier transform yields the FFT algorithm; the semiring is the set of complex numbers with the usual addition and multiplication [6], [10].
- 2) ML decoding of binary linear codes is also an MPF problem and application of SP algorithm yields the Gallager-Tanner-Wiberg decoding algorithm over a Tanner graph; the semiring is the set of positive real numbers with “min” as sum and “+” as product, called the min-sum semiring [6], [10]. The BCJR algorithm for decoding turbo codes and the LDPC decoding algorithm are some other applications of the SP algorithm.
- 3) Application to the ML sequence estimation, for instance in decoding convolutional codes, yields the Viterbi algorithm [10]; the semiring is again the min-sum semiring.
- 4) Recently, the GDL has been shown to reduce the ML decoding complexity of space-time block codes in [9]; the semiring applicable is the min-sum semiring of complex number. The authors introduced traceback for GDL and used it to further lower the number of operations.

Thus, the SP algorithm and the GDL subsume as special cases many well known algorithms.

A. MPF Problems in the Boolean Semiring

The Boolean semiring is the set $\{0, 1\}$ together with the usual Boolean operations \vee (OR) and \wedge (AND). We denote it by $R = (\{0, 1\}, \vee, \wedge)$. The elements 0 and 1 are the *additive* and *multiplicative identities* respectively. The MPF problem defined for this semiring is described below. Let x_1, x_2, \dots, x_n be a collection of variables taking values in finite alphabets

A_1, A_2, \dots, A_n , respectively. For $I = \{i_1, \dots, i_k\} \subseteq [n]$, let $x_I = (x_{i_1}, \dots, x_{i_k})$ and $A_I = A_{i_1} \times \dots \times A_{i_k}$. Let $\mathcal{S} = \{S_1, S_2, \dots, S_M\}$ be a family of M subsets of $[n]$ such that for each $j \in [M]$, there is a function $h_j : A_{S_j} \rightarrow R$. These functions are called the *local functions*, the set of variables in x_{S_j} is called the *local domain* of h_j , and A_{S_j} is the associated *configuration space*. The *global function*, $g : A_{[n]} \rightarrow R$ and the *marginal function*, $g_I : A_I \rightarrow R$, associated with a subset I of $[n]$ are defined as follows:

$$g(x_1, x_2, \dots, x_n) = \bigwedge_{j=1}^M h_j(x_{S_j})$$

and

$$g_I(x_I) = \bigvee_{x_{[n] \setminus I} \in A_{[n] \setminus I}} g(x_1, x_2, \dots, x_n). \quad (1)$$

If we are interested in the support of the marginal $g_I(x_I)$, then the instance x_I^* , if unique, of x_I for which $g_I(x_I^*) = 1$ is obtained as follows:

$$x_I^* = \text{supt } g_I(x_I), \quad (2)$$

and for $J \subseteq I$, x_J^* (if unique) can be obtained as follows:

$$x_J^* = \text{supt}_J g_I(x_I). \quad (3)$$

If the instances of x_I obtained using (2) are not unique, i.e., the support contains more than one I -tuple for which g_I evaluate to 1, then all these can be collected in a set, say, B_I , where $B_I = \text{supt } g_I(x_I) \subseteq A_I$. This is the arg-MPF problem for the Boolean semiring where an instance of some variables (arg) that causes a marginal function (MPF) to evaluate to 1 is required. Other examples of arg-MPF problems include decoding of classical error-correcting codes, ML sequence detection using Viterbi algorithm, and ML decoding of space-time block codes in appropriate min-sum semirings; in all these problem we are interested in obtaining the instance of the variables that cause the marginal functions to evaluate to an element in the semiring which is the least when compared to evaluations at other possible instances of the variables. Similarly, over the Boolean semiring, the instances of a subset of variables which causes some marginal function to take value 0 or 1 may be of interest. When evaluation to 1 is required, we use (2) to obtain such instance(s).

Remark 1: For a binary-valued function f of n variables x_1, \dots, x_n such that $x_i \in A_i$ for all $i \in [n]$, where A_i s are finite alphabets, finding a vector in its support, $\text{supt } f(x_{[n]})$, and outputting OR of all the values it takes for different instances of input variables, $\bigvee_{x_{[n]} \in A_{[n]}} f(x_{[n]})$, can be seen as the same operation with different outputs. If a table of values of f for different instances of $x_{[n]}$ is given, both go through the values in the columns of function values, and when a 1 is encountered for the first time, the former outputs the instance of input variables and the latter outputs 1. Thus, both these operations require same number of comparison which is at most $|A_{[n]}| - 1$.

Remark 2: The number of comparisons required in (3) is independent of J and is at most $|A_I| - 1$.

The Boolean satisfiability problem is an example of the MPF problem over the Boolean semiring [6]. Given a set of M Boolean expressions in n variables, a Boolean satisfiability problem asks whether there exists an assignment of 0 or 1 to the variables such that all the expressions evaluate to 1 simultaneously. For example, let $h_1 = x_1 \vee x_2$ and $h_2 = x_2 \wedge (x_3 \vee x_4)$ be two Boolean expressions in 4 variables and the objective is to determine whether out of 16 ($= 2^4$) possible values of (x_1, x_2, x_3, x_4) there exists one for which both h_1 and h_2 evaluate to 1.

$$h(x_1, x_2, x_3, x_4) = \bigvee_{(x_1, x_2, x_3, x_4) \in \{0,1\}^4} h_1(x_1, x_2) \wedge h_2(x_2, x_3, x_4).$$

The function h evaluates to 1 if there exists one such assignment and 0 otherwise. The function h can be taken as the marginal function of the global function $h_1(x_1, x_2) \wedge h_2(x_2, x_3, x_4)$ associated with the set $\{x_1, x_2, x_3, x_4\}$. The assignment, if unique, that satisfies all the Boolean expressions is $\text{supt } h(x_1, x_2, x_3, x_4)$. If multiple assignments satisfy the expressions, then they can be collected in a set as stated before. These two cases, unique and non-unique solutions, will arise in decoding network codes and in-network function computation problems respectively.

Solving a system of M linear or polynomial equations in n variables, say $p_1(x_{[n]}) = c_1, p_2(x_{[n]}) = c_2, \dots, p_M(x_{[n]}) = c_M$ over a finite field, where c_1, \dots, c_M are constants, can also be posed as an arg-MPF problem over the Boolean semiring as follows:

$$x_{[n]}^* = \text{supt} \bigwedge_{i=1}^M \delta(p_i(x_{[n]}), c_i),$$

where δ is the Kronecker delta function defined as

$$\delta(a, b) = \begin{cases} 0, & \text{if } a \neq b \\ 1, & \text{if } a = b. \end{cases}$$

The local functions are $\delta(p_1(x_{[n]}), c_1), \dots, \delta(p_M(x_{[n]}), c_M)$.

Compared to the MPF problems, the arg-MPF problems requires an additional step of obtaining the desired support set.

B. The SP Algorithm

The SP algorithm is an efficient way of computing the marginal functions (1), which may require $\mathcal{O}(A_{[n]})$ operations if computed in a brute-force manner. It involves iteratively passing *messages* along the edges of a *factor graph*, $G = (V \cup W, E)$, associated with the given MPF problem. Let $Z = V \cup W$. The factor graph is a bipartite graph. Vertices in V are called variable nodes; one for each variable x_i for all $i \in [n]$ ($|V| = n$) and are labeled x_i . The local domain and configuration space associated with a variable node with label x_i are $\{x_i\}$ and A_i respectively. A variable node does not have a local function. The vertices in W are called the factor nodes; one for each local function $h_j(x_{S_j})$ for all $j \in [M]$ ($|W| = M$) and are labeled h_j . For a factor node with label h_j , its local kernel is $h_j(x_{S_j})$, local domain is x_{S_j} , i.e., the set of variables which are its arguments, and the configuration

space is A_{S_j} . A variable node x_i is connected to a factor node h_j iff x_i is an argument of h_j , i.e., $i \in S_j$.

Let $N(x_i)$ denote the set of factor nodes adjacent to the variable node x_i , i.e., set of local functions with x_i as an argument, and $N(h_j)$ denote the set of variable nodes adjacent to the factor node h_j , i.e., the local domain x_{S_j} of h_j . The directed message passed from a variable node x_i to an adjacent factor node h_j and vice versa are as follows:

$$\mu_{x_i \rightarrow h_j}(x_i) = \bigwedge_{h' \in N(x_i) \setminus h_j} \mu_{h' \rightarrow x_i}(x_i) \quad (4)$$

$$\mu_{h_j \rightarrow x_i}(x_i) = \bigvee_{x_{S_j \setminus i} \in A_{S_j \setminus i}} h_j(x_{S_j}) \bigwedge_{x' \in N(h_j) \setminus x_i} \mu_{x' \rightarrow h_j}(x') \quad (5)$$

The messages are actually tables of values containing value of the messages corresponding to different values of their arguments.

Depending on the requirement, we may need to evaluate marginal(s) at only one, a few or all nodes in the factor graph; the versions of SP algorithm applied to these cases are referred to as the single-vertex, multiple-vertex, and all-vertex SP algorithm respectively. In all these cases, all the messages are initially directed to one node, called the root, i.e., all the edges are directed towards the root and the messages are passed along the direction of the edge. The algorithm starts at the leaf nodes (nodes with degree one) with these nodes passing messages to the adjacent nodes. If a leaf node is a variable node, then the message value is 1 (the semiring multiplicative identity) for all possible values the variable can take, i.e., $\mu_{x_i \rightarrow h_j}(x_i) = 1$ for all $x_i \in A_i$, where $h_j(x_i)$ is the unique local function with x_i as an argument. If a leaf node is a factor node, then its local domain will contain only one variable, say x_i , and $\mu_{h_j \rightarrow x_i}(x_i) = h_j(x_i)$ for all $x_i \in A_i$. Once a vertex has received messages from all but one of its neighbors, it computes its own message using (4) or (5), and passes it to the neighbor from which it has not yet received the message. This continues until the root has received messages on all its edges. Now the root computes and passes the messages to its neighbors and the process continues with messages being passed on each edge in the opposite direction, i.e., away from the root. This message passing terminates when all the nodes at which marginals are required to be computed have received messages from all its neighbors. After receiving messages from all its neighbors, a variable node x_i computes its marginal function g_i as follows:

$$g_i(x_i) = \bigwedge_{h' \in N(x_i)} \mu_{h' \rightarrow x_i}(x_i),$$

and the value x_i^* for which $g_i(x_i) = 1$ is

$$x_i^* = \text{supt } g_i(x_i).$$

The marginal function at a factor node h_j can be computed as follows:

$$g_j(x_{S_j}) = h_j(x_{S_j}) \bigwedge_{x' \in N(h_j)} \mu_{x' \rightarrow h_j}(x'),$$

and required supports can be computed using (2) or (3). As stated in Section II-A, if there are multiple instances of an argument for which a marginal function evaluate to 1, then they can be collected in a set.

To obtain the correct value of the required marginal functions, it is essential that the factor graph be free of cycles [6]. If there are cycles, these may not be the correct values and the sets B_i and C_j may contain some undesired instances of arguments for which the marginals take value 0, in addition to the support of the marginals. We use *variable stretching* (refer to [6, Sec. VI-B and C] for a detailed description) to eliminate cycles; this is explained below. Let G be a connected factor graph with cycles, $N(x)$ be the neighbors of a variable node x in G , and let T be a spanning tree of G . Every variable node x is connected to all the factor nodes in $N(x)$ in G but not in T . In T , there is a unique path from every variable node x to the factor nodes in $N(x)$, since it is a tree. For each variable x , add x to the local domains of all the nodes in the aforementioned unique paths; this is referred to as stretching variable x . The resulting factor graph with enlarged local domains is acyclic and is denoted by G' . The SP algorithm applied to G' will give the exact marginal functions [6, Sec. VI-C]. If the factor graph is not connected, then we find a spanning tree of each connected component and perform variable stretching in each of the trees. This method is exemplified below.

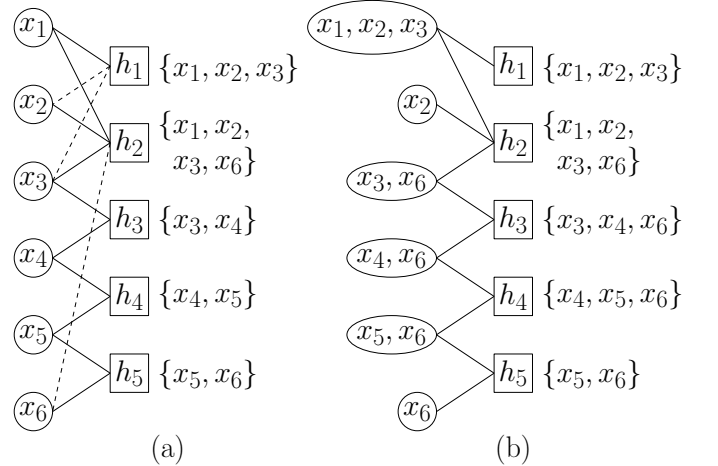


Fig. 2. Variable stretching (a) A factor graph with cycles and (b) its acyclic version after stretching variables x_2 , x_3 , and x_6 along the unique path from x_2 to h_1 , x_3 to h_1 , and x_6 to h_2 respectively in the factor tree obtained by removing dashed edges in (a).

A factor graph with cycles is given in Fig. 2(a) and a spanning tree is obtained by deleting the dashed edges. In Fig. 2(a), $N(x_2) = \{h_1, h_2\}$, but x_2 is not connected to h_1 in the spanning tree and hence is stretched along the unique path $x_2 - h_2 - x_1 - h_1$ in Fig. 2(a) between x_2 and h_1 resulting in the addition of x_2 to the local domain of variable node x_1 (local domain of h_2 already contains x_2). Similarly, x_3 is added to the local domain of variable node x_1 which lies in the unique path $x_3 - h_2 - x_1 - h_1$ between x_3 and h_1 , and x_6 is added to the local domains of x_5 , h_4 , x_4 , h_3 , and x_3 which lie on the unique path $x_6 - h_5 - x_5 - h_4 - x_4 - h_3 - x_3 - h_2$ from x_6 to h_2 . The resulting factor graph is depicted in Fig. 2(b).

The local functions of the factor nodes remain the same and the variable nodes are now labeled by the new enlarged local domains. In the new acyclic factor graph, we denote the variable and factor nodes by v and w and their local domains by S_v and S_w respectively. The SP algorithm on the modified graph proceed as before; message passing starts at the leaf node and terminates when each node has received a message from all its neighbors. The message passed from a variable node v to a factor node w in the new graph is

$$\mu_{v \rightarrow w}(x_{S_v \cap S_w}) = \bigvee_{x_{S_v \setminus S_w} \in A_{S_v \setminus S_w}} \bigwedge_{w' \in N(v) \setminus w} \mu_{w' \rightarrow v}(x_{S_{w'} \cap S_v}), \quad (6)$$

and that passed from a factor node w to a variable node v is

$$\mu_{w \rightarrow v}(x_{S_w \cap S_v}) = \bigvee_{x_{S_w \setminus S_v} \in A_{S_w \setminus S_v}} h_w(x_{S_w}) \bigwedge_{v' \in N(w) \setminus v} \mu_{v' \rightarrow w}(x_{S_{v'} \cap S_w}), \quad (7)$$

where h_w is the local function of factor node w . The marginal function of a variable node v is

$$g_v(x_{S_v}) = \bigwedge_{w' \in N(v)} \mu_{w' \rightarrow v}(x_{S_{w'} \cap S_v}), \quad (8)$$

and that of a factor node w is

$$g_w(x_{S_w}) = h_w(x_{S_w}) \bigwedge_{x' \in N(h_w)} \mu_{x' \rightarrow h_w}(x'). \quad (9)$$

As before, the required supports can be computed using (2) or (3).

From (6)-(9), it can be inferred that the number of operations required to compute messages and marginal functions in the SP algorithm will be $\mathcal{O}(A_{z^*})$, where z^* is the node with the largest configuration space A_{z^*} .

III. DECODING NETWORK CODES USING THE SP ALGORITHM

In this section, we show that decoding a network code is an arg-MPF problem over the Boolean semiring. We provide a method to construct factor graph for decoding at a sink node using the SP algorithm.

Though the factor graph approach [6] and the junction tree approach [10] are equivalent formulations to solve MPF problems, we prefer the former because of the difference in the amount of preprocessing required to obtain a junction tree as argued below:

- 1) The construction of a junction tree for an MPF problem requires [10, Sec. IV]: (a) construction of a *local domain graph* with weighted edges, (b) finding a maximum weight spanning tree, (c) checking whether the sum of edges weights of the obtained maximum weight spanning tree is equal to $\sum_{j=1}^M |S_i| - n$, if yes then this tree is a junction tree for the MPF problem, otherwise we proceed with (d) construction of a *moral graph*, (e) obtaining its *minimum complexity triangulation* if it is not already triangulated, (f) construction of the *clique graph* of the triangulated moral graph, and (g) finding a spanning tree which leads to minimum computational cost. To the nodes of this clique tree, called core in [9], the local functions and variables of the MPF problem are

attached [10] to obtain the junction tree (a local function or a variable node is attached to a node of the core iff its local domain is a subset of the local domain of the said core node). Thus, the GDL always gives the exact solution of the MPF problems.

- 2) A factor graph is described by the local functions associated with the MPF problem. If it is acyclic, then the SP algorithm gives the exact solution, if not, it gives an approximate solution [6]. The SP algorithm is known to perform well even if the factor graph has cycles, for example, in the iterative decoding of LDPC and turbo codes. As explained and exemplified in Section II-B, cycles in a factor graph can be eliminated by first obtaining a spanning tree of the factor graph with cycles and then performing variable stretching [6, Sec. VI-C]. The SP algorithm applied to the new acyclic factor graph will yield the exact marginal functions.

A. Network Code Decoding as an MPF Problem

Given an acyclic network $\mathcal{N} = (\mathcal{V}, \mathcal{E})$, the demands at each sink, D_k , $k \in [K]$ and a set of global encoding maps, $\{\tilde{f}_e : e \in \mathcal{E}\}$, that satisfy all the sink demands, the objectives at a sink, say k^{th} , is to find the instance of x_{D_k} that was generated by the source(s) using the data it receives on its incoming edges, i.e.,

$$x_{D_k}^* = \sup_{D_k} \bigwedge_{e \in In(T_k)} \delta(\tilde{f}_e(x_{[\omega]}), y_e) = \sup_{D_k} g^{(k)}(x_{[\omega]}). \quad (10)$$

Here $g^{(k)}$ is the global function of the MPF problem at the k^{th} sink. For an LNC, $\tilde{f}_e(x_{[\omega]}) = \mathbf{f}_e \cdot \mathbf{x}$.

Thus, decoding a network code has the form of an arg-MPF problem over the Boolean semiring wherein we are interested only in some coordinates (specified by D_k) of the ω -tuples in the support set and not the value of the global function.

Since the solution $x_{D_k}^*$ is unique, individual coordinates $j \in D_k$ can be separately computed, i.e.,

$$x_j^* = \sup_j \bigvee_{x_j \in F} g_j^{(k)}(x_j) \\ g_j^{(k)}(x_j) = \bigvee_{x_{[\omega] \setminus j} \in F^{\omega-1}} g^{(k)}(x_{[\omega]}), \quad (11)$$

where $g_j^{(k)}(x_j)$ is a marginal function of the global function $g^{(k)}$.

The factor graph for decoding at sink T_k , $k \in [K]$ is constructed as follows:

- 1) Install ω *variable nodes*, one for each source message. These vertices are labeled by their corresponding source messages, x_i . The local kernel of these nodes are 1.
- 2) Install $|In(T_k)|$ *factor nodes* and label them \tilde{f}_e , $e \in In(T_k)$. The associated local domain of each such vertex is the subset $S_e \subseteq \{x_{[\omega]}\}$ of source messages that participate in that encoding map and the local function is $\delta(\tilde{f}_e(x_{S_e}), y_e)$.
- 3) A variable node is connected to a factor node iff the source message corresponding to that variable node

participates in the encoding map corresponding to the said factor node.

General form of a factor graph and the same for the two sink nodes of the butterfly network are given in Fig. 3.

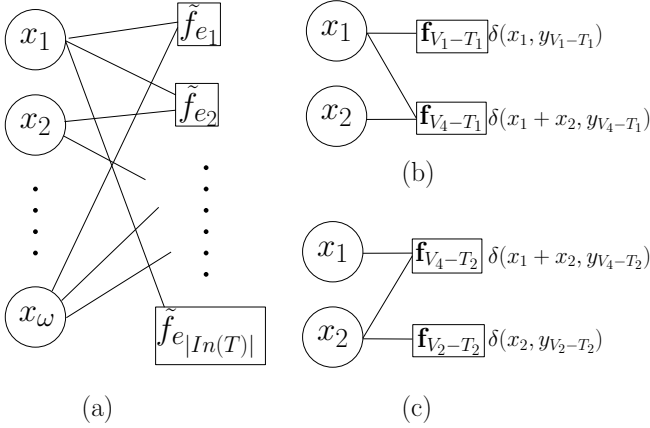


Fig. 3. (a) General form of a factor graph. (b) Factor graphs for T_1 and (c) T_2 of the butterfly network. Local function are given adjacent to the factor nodes.

As specified in Section II-B, the SP algorithm yields the correct value of the source messages if the factor graph is a tree. If not, then the cycles in the factor graph are eliminated via variable stretching on a spanning tree of the factor graph. Messages are computed using (6) and (7), marginals using (8) and (9), and the desired supports using (2) or (3).

B. Traceback

Since decoding network codes is an arg-MPF problem and not an MPF problem, we can use traceback [9] to reduce the number of operations.

We first demonstrate how traceback is used for decoding at a sink which demands all the source messages. If there exists a vertex whose local domain is the entire message set, then all the messages can be obtained by running single-vertex SP algorithm with this node as the root. If not, then assume that the single-vertex SP algorithm is run with a vertex, say r , as the root and the values $x_{S_r}^*$, $S_r \subset [\omega]$, of some source messages have been ascertained. Now, partition the local domain of a neighboring node z , as $x_{S_z} = x_A \cup x_B$, where $A = S_z \setminus S_r$ and $B = S_z \cap S_r$. Since $x_{S_r}^*$ is known, the value x_B^* of x_B for which $g_z(x_A, x_B) = 1$ is also known. Then x_A^* can then be obtained as follows:

$$\begin{aligned} x_A^* &= \text{supt } g_z(x_{S_z}) = \text{supt } g_z(x_A, x_B^*) \\ &= \text{supt } \mu_{r \rightarrow z}(x_B^*) \lambda_z(x_A, x_B^*) \\ &= \text{supt } \lambda_z(x_A, x_B^*), \end{aligned} \quad (12)$$

where

$$\lambda_z(x_A, x_B) = h_z(x_{S_z}) \bigwedge_{u \in N(z) \setminus r} \mu_{u \rightarrow z}(x_{S_u \cap S_z})$$

is the partial marginal computed at z while passing the message $\mu_{z \rightarrow r}(x_B)$ to the root r ; the two are related as

follows:

$$\begin{aligned} \mu_{z \rightarrow r}(x_B) &= \bigvee_{x_A} h_z(x_{S_z}) \bigwedge_{u \in N(z) \setminus r} \mu_{u \rightarrow z}(x_{S_u \cap S_z}) \\ &= \bigvee_{x_A \in F^{|S_z|}} \lambda_z(x_A, x_B), \end{aligned}$$

where $h_z(x_{S_z})$ is the local function if z is a factor node and assumed to be 1 for all $x_{S_z} \in F^{|S_z|}$ if z is a variable node. Thus, with traceback, neither computation of $\mu_{r \rightarrow z}(x_{S_r \cap S_z})$ nor that of marginal function $g_z(x_{S_z})$ is needed for any neighbor z of r .

The traceback step is performed until the values of all the source messages are obtained. This is done by obtaining source message values at a chosen root node r , followed by traceback on its neighbors, then the neighbors of neighbors of r , and so on. This can lead to considerable reduction in number of operations and is exemplified in Section III-C, Example 2.

We now present how traceback is performed in decoding network codes at a sink which demands only a subset of the source messages. Let $D \subset [n]$ denote the demand of a sink node. If there exists a vertex with local domain same as or containing D , then all the desired source messages can be obtained by running single-vertex SP algorithm with this node as the root. If not, then assume that the single-vertex SP algorithm is run with a vertex, say r , as the root. Let z be a neighbor of r with local domain x_{S_z} partitioned as given in Fig. 4.

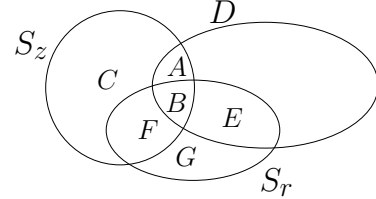


Fig. 4. Venn diagram of the demand set D and local domains of z and r .

Once r has received all the messages, the marginal function $g_r(x_{S_r})$ is computed as follows:

$$g_r(x_{S_r}) = h_r(x_{S_r}) \mu_{z \rightarrow r}(x_B, x_F) \bigwedge_{u \in N(r) \setminus z} \mu_{u \rightarrow r}(x_{S_u \cap S_r}). \quad (13)$$

Since the network code ensures decoding of only (x_B, x_E) , and not (x_F, x_G) , there exists a unique instance of (x_B, x_E) , denoted by (x_B^*, x_E^*) , and multiple instances of (x_F, x_G) , one of which is denoted by (\hat{x}_F, \hat{x}_G) , such that $g_r(x_B^*, x_E^*, \hat{x}_F, \hat{x}_G) = 1$, i.e.,

$$(x_B^*, x_E^*, \hat{x}_F, \hat{x}_G) = \text{supt } g_r(x_B, x_E, x_F, x_G).$$

By Remark 2, computation of $(x_B^*, x_E^*, \hat{x}_F, \hat{x}_G)$ incurs no additional operations over computation of only (x_B^*, x_E^*) ; both require at most $q^{|S_r|} - 1$ comparisons. The message to be passed from r to z is

$$\mu_{r \rightarrow z}(x_B, x_F) = \bigvee_{(x_E, x_G)} h_r(x_{S_r}) \bigwedge_{u \in N(r) \setminus z} \mu_{u \rightarrow r}(x_{S_u \cap S_r}). \quad (14)$$

Since $g_r(x_B^*, x_E^*, \hat{x}_F, \hat{x}_G) = 1$, from (13) we have that

$$h_r(x_{S_r}) \bigwedge_{u \in N(r) \setminus z} \mu_{u \rightarrow r}(x_{S_u \cap S_r}) = 1$$

for $x_{S_r} = (x_B^*, x_E^*, \hat{x}_F, \hat{x}_G)$, and consequently from (14), $\mu_{r \rightarrow z}(x_B^*, \hat{x}_F) = 1$.

Now at z , (x_A^*, \hat{x}_C) is computed as follows:

$$\begin{aligned} (x_A^*, \hat{x}_C) &= \text{supt } g_z(x_A, x_B^*, x_C, \hat{x}_F) \\ &= \text{supt } \mu_{r \rightarrow z}(x_B^*, \hat{x}_F) \lambda_z(x_A, x_B^*, x_C, \hat{x}_F) \\ &= \text{supt } \lambda_z(x_A, x_B^*, x_C, \hat{x}_F), \end{aligned}$$

where $\mu_{r \rightarrow z}(x_B^*, \hat{x}_F) = 1$ as argued above and $\lambda_z(x_{S_z})$ is the partial message computed at z while passing message $\mu_{z \rightarrow r}(x_B, x_F)$ to r ; the two are related as follows:

$$\begin{aligned} \mu_{z \rightarrow r}(x_B, x_F) &= \bigvee_{(x_A, x_C)} h_z(x_{S_z}) \bigwedge_{u \in N(z) \setminus r} \mu_{u \rightarrow z}(x_{S_u \cap S_z}) \\ &= \bigvee_{(x_A, x_C)} \lambda_z(x_{S_z}). \end{aligned}$$

As before, by Remark 2, computation of (x_A^*, \hat{x}_C) incurs no additional operations over computation of only x_A^* ; both require at most $q^{|A|+|C|} - 1$ comparisons.

This process is repeated on other neighbors of r , followed by neighbors of neighbors of r , and so on until values of all the messages in the demand set have been determined.

Thus, for a sink with a general demand set $D \subseteq [\omega]$, in the single-vertex SP algorithm with traceback, first the single-vertex SP algorithm is used with some node (preferably one whose local domain includes some of the demanded messages) as the root wherein all messages are directed towards it, its marginal function is computed, and then the support of the marginal. In the traceback step, appropriate supports of partial marginals, which were already computed while passing the messages towards the root, of some more nodes are computed; this involves only comparison operations. Let the root together with the set of nodes involved in the traceback step, i.e., nodes whose union of local domains contain the demand set, be denoted by Z' .

In the multiple-vertex SP algorithm, computing marginal functions of nodes in Z' is enough since their appropriate support will satisfy the sink's demands. As above, first the single-vertex SP algorithm is used. When the root has received all the messages, it passes messages to its neighbors and message passing continues until all other nodes in Z' have received all the messages. After this, marginal functions of the nodes in Z' is computed, and then appropriate support (only the intersection of local domain of the root and D) of the marginal function is computed.

By Remark 2, computation of supports in both the single-vertex with traceback and in the last step of the multiple-vertex SP algorithm incurs the same computational cost. But the latter involves computation of additional messages (directed away from root) and marginals (of nodes in Z' other than the root). Hence, the traceback step reduces the number of operations required in decoding a network code. We refer to the use of single-vertex SP decoding followed by traceback as the *reduced complexity SP decoding*. Exact number of operations

required in the all-vertex SP algorithm and single-vertex SP algorithm with traceback is derived in Section IV.

C. Illustrations

We now present some examples illustrating use of the SP algorithm to decode network codes.

Example 1: Consider the butterfly network of Fig.1. Here $q = \omega = 2$. The factor graphs for two sink nodes are given in Fig. 3(b) and (c). The messages passed and state computations for decoding at T_1 are as follows:

$$\begin{aligned} \mu_{x_2 \rightarrow f_{V_4-T_1}}(x_2) &= 1, \\ \mu_{f_{V_1-T_1} \rightarrow x_1}(x_1) &= \delta(x_1, y_{V_1-T_1}), \\ \mu_{f_{V_4-T_1} \rightarrow x_1}(x_1) &= \bigvee_{x_2} \delta(x_1 + x_2, y_{V_4-T_1}), \\ \mu_{x_1 \rightarrow f_{V_4-T_1}}(x_1) &= \mu_{f_{V_1-T_1}, x_1}(x_1), \\ g_{x_1}(x_1) &= \mu_{f_{V_1-T_1}, x_1}(x_1) \mu_{f_{V_4-T_1} \rightarrow x_1}(x_1), \\ \mu_{f_{V_4-T_1} \rightarrow x_2}(x_2) &= \bigvee_{x_1} \delta(x_1 + x_2, y_{V_4-T_1}) \mu_{f_{V_1-T_1} \rightarrow x_1}(x_1), \\ g_{x_2}(x_2) &= \mu_{f_{V_4-T_1}, x_2}(x_2), \\ x_1^* &= \text{supt } g_{x_1}(x_1), \\ x_2^* &= \text{supt } g_{x_2}(x_2). \end{aligned}$$

Similar computations apply for T_2 also. \square

In the following example we present a network with general demands at sinks and employ the SP algorithm for decoding a vector nonlinear network code for it. We also demonstrate the usefulness of traceback in saving computations of some messages in the factor graph.

Example 2: Consider the network given in Fig. 5. The sinks (nodes 37 – 46) have general demands which are specified below them. In [11], the authors showed that this network admits no linear solution over any field and gave a vector nonlinear solution. The source messages $x_i, i \in [5]$ are 2-bit binary words ($q = 4, \omega = 5$), $+$ denotes addition in ring \mathbf{Z}_4 , \oplus denotes the bitwise XOR and the function $t(\cdot)$ reverses the order of the 2-bit input.

The factor graphs for nodes 37, 40, and 43, denoted by G_{37} , G_{40} , and G_{43} respectively, are given in Fig. 6. The 4-cycle in G_{40} is removed by deleting the dashed edge and stretching variable x_2 along the unique path P from x_2 to the factor node labeled by $x_1 + x_2 + x_3$. Similarly, the two 6-cycles in G_{43} are removed by deleting dashed edges and stretching variable x_3 along paths P_1 and P_2 ; for convenience, nodes are numbered a-k in the acyclic factor graph.

We infer from G_{43} that the number of computations required to reproduce all the source messages at V_{43} is only $\mathcal{O}(q^3)$ instead of $\mathcal{O}(q^5)$ (as brute-force decoding would have required). The decoding process at V_{43} is performed by using single-vertex SP algorithm with node “i” in G_{43} as the root to compute x_3, x_4 , and x_5 followed by traceback to compute

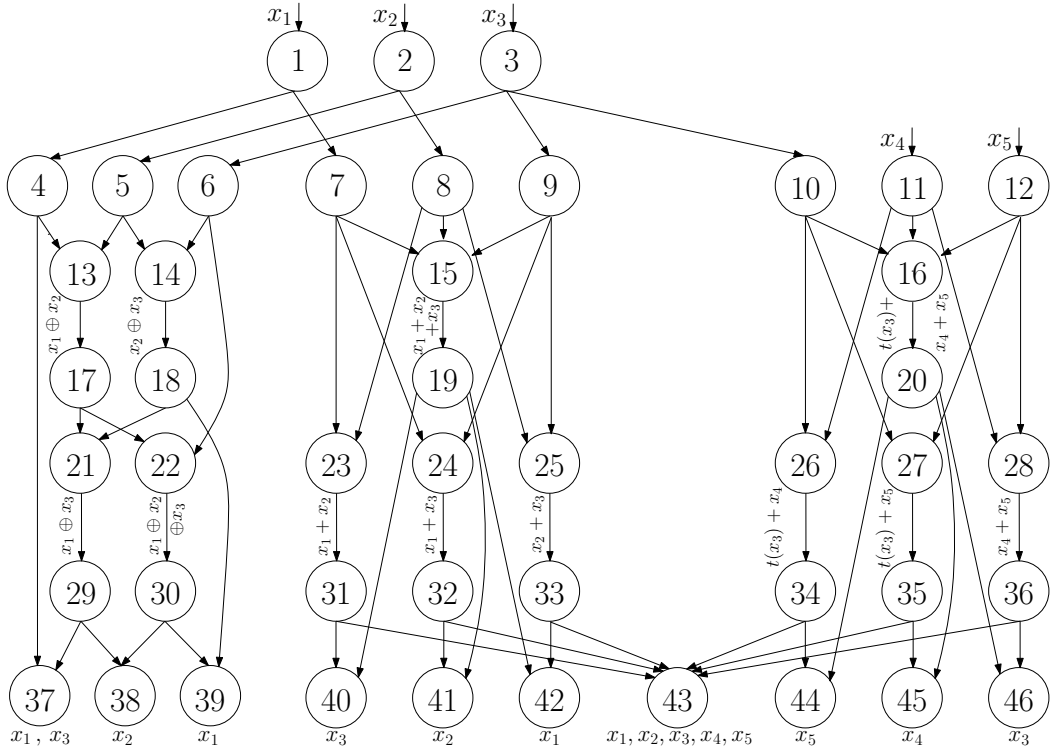


Fig. 5. The network \mathcal{N}_3 of [11].

x_1 and x_2 . The messages passed towards root are

$$\begin{aligned}\mu_{k \rightarrow j} &= \mu_{j \rightarrow i}(x_3, x_5) = \delta(t(x_3) + x_5, y_{35-43}), \\ \mu_{a \rightarrow b} &= \mu_{b \rightarrow c}(x_2, x_3) = \delta(x_2 + x_3, y_{33-43}), \\ \mu_{c \rightarrow d} &= \mu_{d \rightarrow e}(x_1, x_3) = \bigvee_{x_2} \delta(x_1 + x_2, y_{31-43}) \mu_{b \rightarrow c}(x_2, x_3),\end{aligned}$$

$$\mu_{e \rightarrow f} = \mu_{f \rightarrow g}(x_3) = \bigvee_{x_1} \delta(x_1 + x_3, y_{32-43}) \mu_{d \rightarrow e}(x_1, x_3),$$

and

$$\mu_{g \rightarrow h} = \mu_{h \rightarrow i}(x_3, x_4) = \delta(t(x_3) + x_4, y_{34-43}) \mu_{f \rightarrow g}(x_3).$$

Decoding of x_3 , x_4 , and x_5 is performed at “i” by first computing the marginal function F_i using (2) and then computing its support as follows:

$$\begin{aligned}F_i(x_3, x_4, x_5) &= \delta(x_4 + x_5, y_{36-43}) \mu_{h \rightarrow i}(x_3, x_4) \mu_{j \rightarrow i}(x_3, x_5), \\ (x_3^*, x_4^*, x_5^*) &= \text{supt } F_i(x_3, x_4, x_5).\end{aligned}$$

Since x_3^* is known, x_1^* and x_2^* are computed using traceback at nodes “e” and “c” respectively as follows:

$$x_1^* = \text{supt } \lambda_e(x_1, x_3^*)$$

and

$$x_2^* = \text{supt } \lambda_c(x_1^*, x_2, x_3^*),$$

where the partial marginal $\lambda_e(x_1, x_3) = \delta(x_1 + x_3, y_{32-43}) \mu_{d \rightarrow e}(x_1, x_3)$ was computed while passing the message $\mu_{e \rightarrow f}(x_3)$ and $\lambda_c(x_1^*, x_2, x_3^*) = \delta(x_1 + x_2, y_{31-43}) \mu_{b \rightarrow c}(x_2, x_3)$ was computed while passing the message $\mu_{c \rightarrow d}(x_1, x_3)$. In other words,

$$\mu_{e \rightarrow f}(x_3) = \bigvee_{x_1} \lambda_e(x_1, x_3)$$

and

$$\mu_{c \rightarrow d}(x_1, x_3) = \bigvee_{x_2} \lambda_c(x_1, x_2, x_3).$$

The number of semiring operations required to compute all the messages passed and marginals computed are tabulated in Table II.

TABLE II
SINGLE-VERTEX SP ALGORITHM WITH TRACEBACK

		No. of \wedge	No. of \vee
C_1	$\mu_{k \rightarrow j}, \mu_{j \rightarrow i}$	0	0
C_2	$\mu_{a \rightarrow b}, \mu_{b \rightarrow c}$	0	0
C_3	$\mu_{c \rightarrow d}$	q^3	$q^2(q-1)$
C_4	$\mu_{d \rightarrow e}$	0	0
C_5	$\mu_{e \rightarrow f}$	q^2	$q(q-1)$
C_6	$\mu_{f \rightarrow g}$	0	0
C_7	$\mu_{g \rightarrow h}$	q^2	0
C_8	$\mu_{h \rightarrow i}$	0	0
C_9	$F_i(x_3, x_4, x_5)$	$2q^3$	0
C_{10}	(x_3^*, x_4^*, x_5^*)	0	$q^3 - 1$
C_{11}	x_1^*, x_2^*	0	$q - 1$

When not using traceback, SP decoding is performed by computing messages $\mu_{k \rightarrow j}, \mu_{j \rightarrow i}, \mu_{a \rightarrow b}, \mu_{b \rightarrow c}, \mu_{c \rightarrow d}, \mu_{d \rightarrow e}, \mu_{e \rightarrow f}, \mu_{f \rightarrow g}, \mu_{g \rightarrow h}, \mu_{h \rightarrow i}$ as before and then messages $\mu_{i \rightarrow h}$,

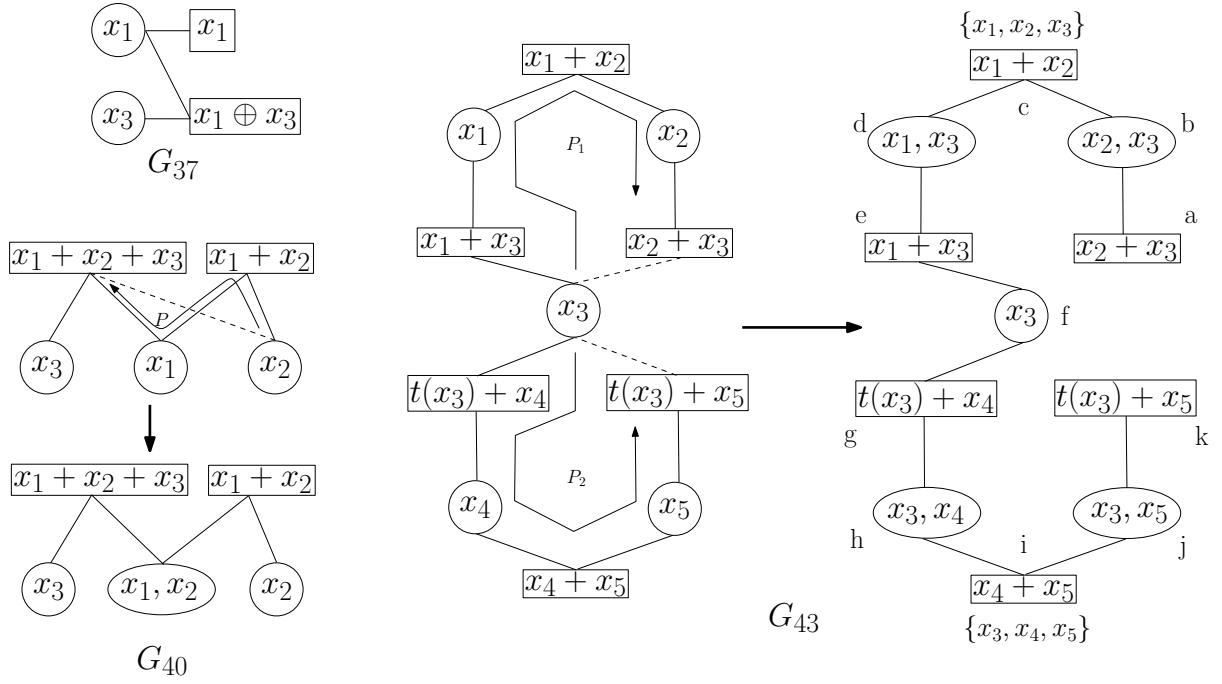


Fig. 6. The factor graphs for sinks with vertex labels 37, 40 and 43 in network \mathcal{N}_3 of Fig. 5. Enlarged local domains after variable stretching are given adjacent to factor nodes.

$\mu_{h \rightarrow g}$, $\mu_{g \rightarrow f}$, $\mu_{f \rightarrow e}$, $\mu_{e \rightarrow d}$, and $\mu_{d \rightarrow c}$ are computed as follows:

$$\mu_{i \rightarrow h} = \mu_{h \rightarrow g}(x_3, x_4) = \bigvee_{x_5} \delta(x_4 + x_5, y_{36-43}) \mu_{j \rightarrow i}(x_3, x_5),$$

$$\mu_{g \rightarrow f} = \mu_{f \rightarrow e}(x_3) = \bigvee_{x_4} \delta(t(x_3) + x_4, y_{34-43}) \mu_{h \rightarrow g}(x_3, x_4),$$

and

$$\mu_{e \rightarrow d} = \mu_{d \rightarrow c}(x_1, x_3) = \mu_{f \rightarrow e}(x_3) \delta(x_1 + x_3, y_{32-43}).$$

At “i”, x_3^* , x_4^* , and x_5^* are obtained as given above, and x_1 and x_2 are obtained at “e” and “c” respectively by first computing the marginal functions and then their appropriate supports using (3) as follows:

At “e”

$$F_e(x_1, x_3) = \delta(x_1 + x_3, y_{32-43}) \mu_{f \rightarrow e}(x_3),$$

$$x_1^* = \text{supt}_{x_1} F_e(x_1, x_3),$$

and at “c”

$$F_c(x_1, x_2, x_3) = \delta(x_1 + x_2, y_{31-43}) \mu_{d \rightarrow c}(x_1, x_3),$$

$$x_2^* = \text{supt}_{x_2} F_c(x_1, x_2, x_3).$$

The number of semiring operations required to compute additional messages and marginals are tabulated in Table III.

Total number of operations (ANDs and ORs) required with traceback is $2C_1 + 2C_2 + C_3 + \dots + C_{10} + 2C_{11}$, which is $5q^3 + 2q^2 + q - 3 = 353$ operations, and that without traceback are $2C_1 + 2C_2 + C_3 + \dots + C_{10} + C_{12} + \dots + C_{21}$, which is $9q^3 + 6q^2 - 2q - 3 = 661$ operations. Thus, running single-vertex SP algorithm followed by traceback step affords computational advantage over the multiple-vertex version. \square

TABLE III
MULTIPLE-VERTEX SP ALGORITHM

		No. of \wedge	No. of \vee
C_{12}	$\mu_{i \rightarrow h}$	q^3	$q^2(q-1)$
C_{13}	$\mu_{h \rightarrow g}$	0	0
C_{14}	$\mu_{g \rightarrow f}$	q^2	$q(q-1)$
C_{15}	$\mu_{f \rightarrow e}$	0	0
C_{16}	$\mu_{e \rightarrow d}$	q^2	0
C_{17}	$\mu_{d \rightarrow c}$	0	0
C_{18}	$F_e(x_1)$	q^2	0
C_{19}	x_1^*	0	$q^2 - 1$
C_{20}	$F_c(x_2)$	q^3	0
C_{21}	x_2^*	0	$q^3 - 1$

IV. COMPLEXITY OF THE SP ALGORITHM

We will now determine the number of semiring operations required to compute the desired marginal functions in an MPF problem using the SP algorithm and the desired supports in an arg-MPF problem using the arg-SP algorithm with and without traceback in the Boolean semiring.

In this section, by addition and multiplication we mean the Boolean OR and AND operations. By Remark 1, supt is considered same as addition. Let $G = (Z, E) = (V \cup W, E)$ be an acyclic factor graph with variable nodes V and factor node W . The local domain of a node z is denoted by x_{S_z} , the cardinality of its configuration space A_{S_z} by q_z , and its degree by d_z . For an edge $e = (a, b)$ between nodes a and b , $q_e = q_{a \cap b} = |A_{S_a \cap S_b}|$ and $q_{a \setminus b} = |A_{S_a \setminus S_b}|$. For every node $z \in Z$, define $a_z = 1$ if $z \in W$ and 0 otherwise.

A. Single-vertex SP and arg-SP Algorithms

The message passed from a variable node v to a factor node w as given in (6) is

$$\mu_{v \rightarrow w}(x_{S_v \cap S_w}) = \bigvee_{x_{S_v \setminus S_w} \in A_{S_v \setminus S_w}} \bigwedge_{w' \in N(v) \setminus w} \mu_{w' \rightarrow v}(x_{S_{w'} \cap S_v}).$$

In the above equation, for each of the $q_{v \setminus w}$ values of $x_{S_v \setminus S_w}$, product of $d_v - 1$ messages is required which requires $d_v - 2$ multiplications. For each of the $q_{v \cap w}$ values of $x_{S_v \cap S_w}$, $q_{v \setminus w} - 1$ additions and $q_{v \setminus w}(d_v - 2)$ multiplications are required. Thus, the total number of operations required are

$$\begin{aligned} q_{v \cap w}(q_{v \setminus w} - 1) &= q_v - q_{v \cap w} \text{ additions and} \\ q_{v \cap w} q_{v \setminus w}(d_v - 2) &= q_v(d_v - 2) \text{ multiplications.} \end{aligned}$$

The messages passed from a factor node w to a variable node v as given in (7) is

$$\mu_{w \rightarrow v}(x_{S_w \cap S_v}) = \bigvee_{x_{S_w \setminus S_v} \in A_{S_w \setminus S_v}} h_w(x_{S_w}) \bigwedge_{v' \in N(w) \setminus v} \mu_{v' \rightarrow w}(x_{S_{v'} \cap S_w}).$$

This involves product of a local functions with $d_w - 1$ messages for each of the $q_{w \setminus v}$ values of $x_{S_w \setminus S_v}$. The total number of operations required for this case is

$$\begin{aligned} q_{w \cap v}(q_{w \setminus v} - 1) &= q_w - q_{w \cap v} \text{ additions and} \\ q_{w \cap v} q_{w \setminus v}(d_w - 1) &= q_w(d_w - 1) \text{ multiplications.} \end{aligned}$$

The messages are passed by all nodes except the root node. At the root node r , the marginal function is the product of d_r messages, requiring $(d_r - 1)q_r$ multiplications, if it is a variable node (8) and the product of d_r messages with the local function, requiring $d_r q_r$ multiplications, if it is a factor node (9). In other words, computation of marginal function at r requires $(d_r + a_r - 1)q_r$ multiplications. Thus, the total number of additions and multiplications required in the single-vertex SP algorithm is

$$\sum_{v \in V \setminus r} (q_v - q_{v \cap w}) + \sum_{w \in W \setminus r} (q_w - q_{w \cap v}) = \sum_{z \in Z \setminus r} q_z - \sum_{e \in E} q_e,$$

and

$$\begin{aligned} \sum_{v \in V \setminus r} q_v(d_v - 2) + \sum_{w \in W \setminus r} q_w(d_w - 1) + (d_r + a_r - 1)q_r \\ = \sum_{z \in Z} (d_z - 1)q_z - \sum_{v \in V \setminus r} q_v + a_r q_r. \end{aligned}$$

The grand total of the number of additions and multiplications is

$$C_1 = \sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e - \sum_{v \in V} q_v.$$

In the arg-SP algorithm, support of marginal at r is computed which requires $q_r - 1$ additions (by Remark 2) so that the grand total of operations in this case is

$$\begin{aligned} C_2 &= \sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e - \sum_{v \in V} q_v + (q_r - 1) \\ &= C_1 + (q_r - 1). \end{aligned}$$

B. Single-vertex arg-SP Algorithm with Traceback

In this case, first the single-vertex arg-SP algorithm with r as the root is executed on the factor graph. Then the local domain x_{S_z} of a neighbor z of r is partitioned into sets $x_I = x_{S_z \setminus S_r}$ and $x_J = x_{S_z \cap S_r}$. The value x_J^* is already known from decoding at r , and x_I^* is computed using (12) as follows:

$$x_I^* = \text{supt } \lambda_z(x_I, x_J^*),$$

where the table of values of the partial marginal $\lambda_z(x_I, x_J)$ was already computed at z while passing the message $\mu_{z \rightarrow r}(x_J)$ to the root r . We need to look only at the rows for which $x_J = x_J^*$ and output the value of x_I for which $\lambda_z(x_I, x_J^*) = 1$. This requires $q_I - 1 < q_z - 1$ additions, where $x_I \in A_I$ and $q_I = |A_I|$. The total number of multiplications remains the same as in the single-vertex arg-SP algorithm, which is $\sum_{z \in Z} (d_z - 1)q_z - \sum_{v \in V \setminus r} q_v + a_r q_r$, but the number of additions is the sum of the number of additions required in single-vertex SP algorithm ($\sum_{z \in Z \setminus r} q_z - \sum_{e \in E} q_e$) and the number of additions required at each node, which is at most $\sum_{z \in Z} q_z - 1$. Thus, the grand total of operations is at most

$$\begin{aligned} C_3 &= C_1 + \sum_{z \in Z} (q_z - 1) = C_2 + \sum_{z \in Z \setminus r} (q_z - 1) \\ &= \sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e + \sum_{v \in V} q_v + \sum_{z \in Z} (q_z - 1) \\ &= \sum_{z \in Z} d_z q_z - \sum_{e \in E} q_e + \sum_{w \in W} q_w - |Z|. \end{aligned}$$

C. All-vertex SP and arg-SP Algorithms

In the all-vertex SP algorithm, first the messages are passed by all the nodes on the unique path towards the root. When the root has received messages from all its neighbors, messages are passed on each edge in the reverse direction, i.e., away from the root and towards the leaves. When all the leaves have received the messages, marginal functions of each node is computed. We use the method suggested in [10, Sec. V] to compute messages and marginal function.

Let a node z have degree d and has received messages from all but one of its neighbors z' which is on the unique path from z to the root. For an instance x'_{S_z} of x_{S_z} , let k_2, k_3, \dots, k_d be the values of the known messages, k_1 be the value of the message it is yet to receive from y_1 , and h_z be the value of its local function, assumed to be 1 if $z \in V$, i.e., $k_i = \mu_{y_i \rightarrow z}(x'_{S_z})$, $y_i \in N(z)$. The messages \hat{k}_i involves the product of h_z with all k_j s excluding k_i and summing over suitable variables as in (6) and (7); there are d such messages to be sent, one to each neighbor. This can be achieved by computing the following products consecutively: $c_d = h_z k_d$, $c_{d-1} = c_d k_{d-1} = h_z k_{d-1} k_d$, \dots , $c_3 = c_4 k_3 = h_z k_3 k_4 \dots k_d$, $c_2 = c_3 k_2 = h_z k_2 k_3 \dots k_d$; this step requires $d - 2 + a_z$ multiplications. Now z passes $\hat{k}_1 = c_2$ to y_1 (after summing over suitable variables) and awaits the reception of k_1 from z' . Once k_1 is received, the marginal functions is computed, $g_z = k_1 c_2 = h_z k_1 k_2 \dots k_d$, which requires 1 multiplication. Then the following products are computed consecutively: $b_1 = k_1$, $b_2 = b_1 k_2 = k_1 k_2$, $b_3 = b_2 k_3 = k_1 k_2 k_3$, \dots , $b_{d-1} = b_{d-2} k_{d-1} = k_1 k_2 \dots k_{d-1}$; this step requires

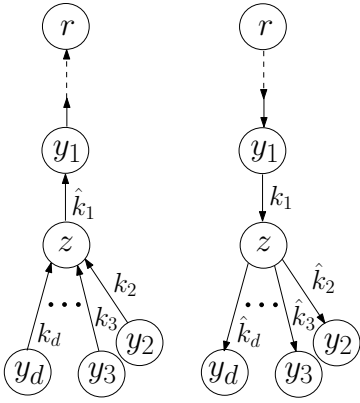


Fig. 7. Message passing at z .

$d - 2$ multiplications. Subsequently, \hat{k}_i s are computed as follows: $\hat{k}_2 = b_1 c_3 = h_z k_1 k_3 k_4 \dots k_d$, $\hat{k}_3 = b_2 c_4 = h_z k_1 k_2 k_4 \dots k_d$, \dots , $\hat{k}_{d-1} = b_{d-2} c_d = h_z k_1 k_2 \dots k_{d-2} k_d$, $\hat{k}_d = h_z b_{d-1} = h_z k_1 k_2 \dots k_{d-1}$; this step requires $d - 2 + a_z$ multiplication. Various messages received and passed by node z are depicted Fig. 7. Thus, computation of all the messages to be passed by z and its marginal function requires $(d - 2 + a_z) + 1 + (d - 2) + (d - 2 + a_z) = 3(d - 2) + 2a_z + 1$ multiplications for each of the q_z values in A_{S_z} . This is true for the root node also. Hence, total number of multiplications required is $\sum_{z \in Z} q_z [3(d - 2) + 2a_z + 1]$. The number of additions required for computing each message remains the same as in the single-vertex SP algorithm, $q_v - q_{v \cap w}$ for a variable node passing message to w . Unlike in the single-vertex case, now v will pass messages to all its d_v neighbors, thus requiring $d_v q_v - \sum_{e \text{ incident on } v} q_e$. Same is true for all the factor nodes also. Hence, the total number of additions required is $\sum_{z \in Z} d_z q_z - 2 \sum_{e \in E} q_e$. The grand total number of operations is then

$$\begin{aligned} C_4 &= \sum_{z \in Z} q_z (3d_z - 5 + 2a_z) + \sum_{z \in Z} d_z q_z - 2 \sum_{e \in E} q_e \\ &= \sum_{z \in Z} (4d_z - 5) q_z + 2 \sum_{w \in W} q_w - 2 \sum_{e \in E} q_e. \end{aligned}$$

In the arg-SP algorithm, computation of support of marginal function at a node z requires at most $q_z - 1$ additions. Thus, the total number of operations required in all-vertex arg-SP algorithm is

$$C_5 = C_4 + \sum_{z \in Z} (q_z - 1) = C_4 + \sum_{z \in Z} q_z - |Z|.$$

The results of Sections IV-A,B, and C are tabulated in Table I. The operation counts presented in this section apply not only to MPF and arg-MPF problem in Boolean semiring, but also to MPF and arg-MPF problem in min-sum, min-product, max-sum, and max-product semiring.

D. Utility and Complexity of SP Algorithm for Decoding Network Code

The SP algorithm for decoding a network code is advantageous when the code is either nonlinear or it is linear but the number of messages is very large. For linear network

codes with manageable value of ω , Gaussian elimination with backward substitution is advisable.

For a node that demands all the source messages, for example a sink in a multicast network, if application of SP algorithm for decoding network codes leads to computational complexity strictly better than the brute-force decoding complexity, then the code is called a *fast SP decodable network code*. The network code for network \mathcal{N}_3 given in Fig. 5 is fast SP decodable for the sink with vertex label 43; decoding complexity is only $\mathcal{O}(q^3)$ compared to the brute-force complexity of $\mathcal{O}(q^5)$.

As stated above, in order to recover the requisite source messages at a sink we need only run the single-vertex arg-SP algorithm followed by traceback steps. For a given sink node, if the factor graph constructed using the method given in Section III-A is cycle-free and the network code is such that the local domains of all factor nodes have cardinality at most $l (< \omega)$, then the number of operations required for decoding using the SP algorithm is $\mathcal{O}(q^l)$. If the sink demands all the source messages, then the brute-force decoding would require $\mathcal{O}(q^\omega) (> \mathcal{O}(q^l))$ operations. Thus, an acyclic factor graph with at most $l (< \omega)$ variables per equation is a sufficient condition for fast decodability of the network code at a sink which demands all the source messages.

If the graph is not cycle-free then we remove the cycles by variable stretching. Let $m \leq \omega$ be the size of maximum cardinality local domain in the new cycle-free factor graph. The number of computations required now will be $\mathcal{O}(q^m) \leq \mathcal{O}(q^\omega)$ and the code is fast decodable iff $m < \omega$.

V. IN-NETWORK FUNCTION COMPUTATION USING THE SP ALGORITHM

A. Preliminaries

In a communication network, some nodes may be interested not in the messages generated by some other nodes but in one or more functions of messages generated by other nodes. For example, in a wireless sensor network that comprises several sensor nodes, each measuring environmental parameters like ambient light, temperature, pressure, humidity, wind velocity etc. For long-term record-keeping and weather forecasting, average, minimum, maximum and variance of these meteorological parameters are of interest. Environmental monitoring in an industrial unit is another field of application where relevant parameter may include temperature and level of exhaust gases which may assist in preventing fire and poisoning due to toxic gases respectively.

We consider in-network function computation in a finite directed acyclic error-free network, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where codes can perform network coding. For brevity of expression, we use x for $x_{[\omega]}$ in this section. The network model is same as given in Section I-A for network coding problem with the exception that the sink nodes demand a function of messages rather than a subset of messages, i.e., a sink node T_k demands the function $g_k : F^\omega \rightarrow F$. A network code comprises global encoding maps $\tilde{f}_e : F^\omega \rightarrow F$, one for each edge $e \in E$, such that there exist K (decoding) maps, $\mathcal{D}_k : F^{|In(T_k)|} \rightarrow F$, for each sink T_k , $k \in [K]$, such that $\mathcal{D}_k(y_e : e \in In(T_k)) = g_k(x)$. This subsumes the network coding problem of Section I as a special

case. By $(y_e : e \in \text{In}(T_k))$ we denote the $|\text{In}(T_k)|$ -tuple of coded messages received by T_k on its incoming edges.

Remark 3: Though arguments of a demanded function g may only be a subset, say $\{x_I\}$ for some $I \subseteq [K]$, of messages, we assume it to be a map from F^ω to F for simplicity rather than from $F^{|I|}$ to F .

Remark 4: If a sink demands $N (> 1)$ functions, then such a sink may be replaced by N sinks each demanding one function but the incoming information to these new sinks is the same (see Fig. 8).

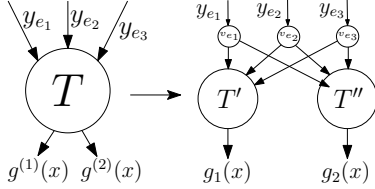


Fig. 8. Converting a sink that demands multiple functions into multiple sinks each with single demand.

The in-network function computation problem is to design network code that maximizes the frequency of target functions computation, called the *computing capacity*, per network use. In [12], bounds on rate of computing symmetric functions (invariant to argument permutations), like minimum, maximum, mean, median and mode, of data collected by sensors in a wireless sensor network at a sink node were presented. The notion of min-cut bound for the network coding problem [1] was extended to function computation problem in a directed acyclic network with multiple sources and one sink in [13]. The case of directed acyclic network with multiple sources, multiple sinks and each sink demanding the sum of source messages was studied in [14]; such a network is called a sum-network. Relation between linear solvability of multiple-unicast networks and sum-networks was established. Furthermore, insufficiency of scalar and vector linear network codes to achieve computing capacity for sum-networks was shown. Coding schemes for computation of arbitrary functions in directed acyclic network with multiple sources, multiple sinks and each sink demanding a function of source messages were presented in [15]. In [16], routing capacity, linear coding capacity and nonlinear coding capacity for function computation in a multiple source single sink directed acyclic network were compared and depending upon the demanded functions and alphabet (field or ring), advantage of linear network coding over routing and nonlinear network coding over linear network coding was shown.

In order to obtain the value of its desired functions, a sink node may require to perform some operations on the messages it receives on the incoming edges. Though there are many results on bounds on the computing capacity and coding schemes for in-function computation problem, the decoding operation to be performed at the sink nodes to obtain the value of the desired functions has not been studied. We now formulate computation of the desired functions at sink nodes as an MPF problem over the Boolean semiring and use the SP algorithm on a suitably constructed factor graph for each sink to obtain the value of the desired functions.

B. Function Computation as an MPF Problem

We consider decoding at the sink node T_k . It demands the function $g_k(x_{I_k})$, where $\{x_{I_k}\} = \{x_{i_1}, x_{i_2}, \dots, x_{i_{|I_k|}}\}$ is the set of arguments of g_k for some $I_k \subseteq [K]$. For a realization x^* of the message vector, we are interested in the value $G_k^* = g_k(x_{I_k}^*)$. Since a network code only ensures computation of the correct value G_k^* of the demanded target function given the incoming coded message vector $(y_e : e \in \text{In}(T_k))$ and not the realization $x_{I_k}^*$ of the messages in the argument set, there may be multiple $|I_k|$ -tuples that produce the same values of the incoming coded messages and function value when input to the demanded function, i.e., the network code is a many-to-one mapping. We denote one such message vector by \hat{x}_{I_k} . It need not necessarily be equal to $x_{I_k}^*$ but $f_e(\hat{x}_{I_k}) = f_e(x_{I_k}^*)$ for all $e \in \text{In}(T_k)$ and $g_k(\hat{x}_{I_k}) = g_k(x_{I_k}^*)$. Using the SP algorithm, we will first obtain \hat{x}_{I_k} and then evaluate $g(\hat{x}_{I_k})$ to obtain G_k^* . The arg-MPF formulation for obtaining \hat{x}_{I_k} is given below. Let

$$S_k = \sup_{I_k} \bigwedge_{e \in \text{In}(T_k)} \delta(\tilde{f}_e(x), y_e) = \sup_{I_k} \beta^{(k)}(x) \quad (15)$$

Here $\beta^{(k)}$ is the global product function and $\delta(\tilde{f}_e(x), y_e)$ are the local functions of the MPF problem at the sink T_k . The set S_k contains the coordinates indexed by I_k of the message vectors x for which $\beta^{(k)}(x) = 1$, i.e., the coordinates indexed by I_k of all those message vectors for which $\tilde{f}_e(x) = y_e$, for all $e \in \text{In}(T_k)$. Though \sup_{I_k} may output multiple $|I_k|$ -tuples, we will choose any one as \hat{x}_{I_k} . The desired function values is then

$$G_k^* = g_k(\hat{x}_{I_k}) \quad (16)$$

Thus, the function computation can be performed by using SP algorithm to solve MPF problem in (15) followed by (16).

Theorem 1: For all $s \in S_k$ obtained using (15) and each $k \in [K]$, we have $g_k(s) = g_k(x_{I_k}^*)$.

Proof: By Remark 3, $g_k(x_{I_k}) = g_k(x)$. A look-up table (LUT) approach to decoding is to maintain a table with q^ω rows and two columns at each sink: first column containing all possible incoming message vectors, $\{(\tilde{f}_e(x) : e \in \text{In}(T_k)) : x \in F^\omega\}$, and the second column listing corresponding values of the demanded function, $\{g_k(x) : x \in F^\omega\}$. Given an instance of incoming messages, a sink node locates the row containing that $|\text{In}(T_k)|$ -tuple in the first column of the LUT and then outputs the value in the second column of the row, which is the desired function value. If two rows in the LUT have the same entry in the first column (network code is a many-to-one map), the entry in the second column will also be same. On the contrary, if for two $x \neq x'$, $g_k(x) \neq g_k(x')$ but $\tilde{f}_e(x) = \tilde{f}_e(x')$ for all $e \in \text{In}(T_k)$ and some $k \in [K]$, then there will be ambiguity at the k th receiver because there are two distinct possible function values, $g_k(x)$ and $g_k(x')$, that the decoder may output.

Thus, a valid network code that fulfills all receivers' demands satisfies $\tilde{f}_e(x) \neq \tilde{f}_e(x')$ for all $e \in \text{In}(T_k)$ if $g_k(x) \neq g_k(x')$ for each $k \in [K]$ and $x \neq x'$, $x, x' \in F^\omega$.

Let x^* be a realization of the message vector and $(y_e : e \in \text{In}(T_k))$ the coded message received by T_k on its incoming

edges. The set

$$S'_k = \sup_{e \in \text{In}(T_k)} \bigwedge \delta(\tilde{f}_e(x), y_e)$$

contains all the message vectors $s' \in F^\omega$ such that $(\tilde{f}_e(s') : e \in \text{In}(T_k)) = (y_e : e \in \text{In}(T_k))$ including x^* . Thus, $g_k(s') = g_k(x^*)$ for all $s' \in S'_k$. Since $S_k = \{s'_{I_k} : s' \in S'_k\}$ and $g_k(x_{I_k}) = g_k(x)$, we have that $g_k(s) = g_k(x^*)$ for all $s \in S_k$. ■

Hence, the SP algorithm for (15) can terminate as soon as a message vector \hat{x}_{I_k} with $\beta^{(k)}(\hat{x}_{I_k}) = 1$ is found and we need not obtain all possible message vectors which evaluate to the given coded messages on incoming edges of a sink.

Example 3: For example, let $\omega = 4$, $x_i \in \mathbb{F}_2$ for all $i \in [4]$, and $g(x_1, x_2, x_3) = x_1 + x_2 + x_3 + \text{Maj}(x_1, x_2, x_3)$ needs to be evaluated using $\tilde{f}_{e_1} = x_1 + x_2$, $\tilde{f}_{e_2} = x_2 + x_3$, and $\tilde{f}_{e_3} = x_1 + x_3$. Here $I = \{1, 2, 3\}$. Let $x^* = 1110$ be a realization of the message vector. Then, $y_{e_1} = 0$, $y_{e_2} = 0$, $y_{e_3} = 0$, and $g(x^*) = 0$. From (15), we have

$$S = \sup_I \bigwedge_{j \in [3]} \delta(\tilde{f}_{e_j}(x), y_{e_j}) = \{000, 111\}$$

Any element of S can be chosen as \hat{x}_I and both evaluate to 0 when input to $g(x_I)$. This illustrates that $g(\hat{x}_I) = g(x^*)$. □

The factor graph for computation of function $g_k(x_{I_k})$ at sink T_k , $k \in [K]$ is constructed as follows:

- 1) Install ω variable nodes, one for each source message. These vertices are labeled by their corresponding source messages, x_i .
- 2) Install $|\text{In}(T_k)|$ factor nodes and label them $\tilde{f}_e, e \in \text{In}(T_k)$. The associated local domain of each such vertex is the set of source messages that participate in that encoding map and the local kernel is $\delta(\tilde{f}_e(x), y_e)$.
- 3) A variable node is connected to a factor node iff the source message corresponding to that variable node participates in the encoding map corresponding to the said factor node.
- 4) Install an additional dummy factor node with local domain $\{x_{I_k}\}$, local kernel 1 and label it g_k . Connect this node to variable nodes in the set $\{x_{I_k}\}$, i.e., to the arguments of g_k . This node corresponds to the demanded function.

As before, first the cycles in the factor graph are removed, if there are any. The single-vertex SP algorithm is run on the acyclic factor graph with the dummy factor node as the root using (6) and (7). Once it has received all the messages, its marginal function (using (9)) and subsequently the set S_k are computed as follows:

$$S_k = \sup_{v \in N(g_k)} \bigwedge \mu_{v \rightarrow g_k}(x_{S_v \cap I_k}),$$

where S_v is the local domain of a neighboring variable node v of g_k . Theorem 1 states that obtaining only an element \hat{x}_{I_k} of the set S_k is sufficient to get the desired function value $g_k^* = g_k(\hat{x}_{I_k})$.

VI. DISCUSSION

In this paper, we proposed to use the SP algorithm for decoding network codes and performing in-network function computation. We posed the problem of network code decoding at each sink node in a network as an MPF problem over the Boolean semiring. A method for constructing a factor graph for a given sink node using the global encoding maps (or vectors in case of an LNC) of the incoming edges and demands of the sink was provided. The graph so constructed had fewer nodes and led to fewer message being passed lowering the number of operations as compared to the scheme of [7]. We discussed the advantages of traceback over multiple-vertex SP algorithm. The number of semiring operations required to perform the SP algorithm with and without traceback were derived. For the sinks demanding all the source messages, we introduced the concept of fast decodable network codes and provided a sufficient condition for a network code to be fast decodable. Then we posed the problem of function computation at sink nodes in an in-network function computation problem as an MPF problem and provided a method to construct a factor graph for each sink node on which SP algorithm can be run to solve the MPF problem.

Using the SP algorithm to decode network error correcting codes is a possible direction of future work.

REFERENCES

- [1] R. W. Yeung, *Information Theory and Network Coding*. New York, NY, USA: Springer, 2008.
- [2] D. S. Lun, M. Médard, R. Koetter, and M. Effros, "On Coding for Reliable Communication Over Packet Networks," *Physical Communication I*, vol. 1, no. 1, pp. 3-20, March 2008.
- [3] D. S. Lun, M. Médard, R. Koetter, and M. Effros, "Further Results on Coding for Reliable Communication Over Packet Networks," in *Proc. IEEE Int. Symp. Information Theory*, 2005, pp. 1848-1852.
- [4] P. G. Ciarlet, *Introduction to Numerical Linear Algebra and Optimization*, New York, NY, USA: Cambridge University Press, 1989.
- [5] G. Strang, *Introduction to Linear Algebra*, 3rd ed., Wellesley, MA, USA: Wellesley-Cambridge Press, 2003.
- [6] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor Graphs and the Sum-Product Algorithm," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 498-519, February 2001.
- [7] D. Salmond, A. Grant, T. Chan, and I. Grivell, "Decoding Network Code by Message Passing," in *Proc. IEEE Int. Symp. Information Theory*, 2009, pp. 423-427.
- [8] M. Mézard and A. Montanari, *Information, Physics, and Computation*, Oxford, NY, USA: Oxford University Press, 2009.
- [9] L. P. Natarajan and B. S. Rajan, "Generalized Distributive Law for ML Decoding of Space-Time Block Codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 2914-2935, May 2013.
- [10] S. M. Aji and R. J. McEliece, "The Generalized Distributive Law," *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 325-343, March 2000.
- [11] R. Dougherty, C. Freiling, and K. Zeger, "Insufficiency of Linear Coding in Network Information Flow," *IEEE Trans. Inf. Theory*, vol. 51, no. 8, pp. 2745-2759, August 2005.
- [12] A. Giridhar and P. R. Kumar, "Computing and Communicating Functions Over Sensor Networks," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 4, pp. 755-764, April 2005.
- [13] R. Appuswamy, M. Franceschetti, N. Karamchandani, and K. Zeger, "Network Coding for Computing: Cut-set Bounds," *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 1015-1030, February 2011.
- [14] B. K. Rai and B. K. Dey, "On Network Coding for Sum-Networks," *IEEE Trans. Inf. Theory*, vol. 58, no. 1, pp. 50-63, January 2012.
- [15] V. Shah, B. K. Dey, and D. Manjunath, "Network Flows for Function Computation," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 4, pp. 714-730, April 2013.
- [16] R. Appuswamy, M. Franceschetti, N. Karamchandani, and K. Zeger, "Linear Codes, Target Function Classes, and Network Computing Capacity," *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 5741-5753, September 2013.